

UNIVERSITY OF PENNSYLVANIA

DOUBLE ELLIPTIC GEOMETRY IN TERMS
OF POINT, ORDER AND CONGRUENCE

DONALD A. FLANDERS

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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PRINTED IN GERMANY

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Reprint from

THE ANNALS OF MATHEMATICS, Vol. 28, No. 4

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DOUBLE ELLIPTIC GEOMETRY IN TERMS OF POINT, ORDER AND CONGRUENCE.*

BY DONALD A. FLANDERS.

1. **Introduction.** The set of axioms presented in this paper possesses two properties of particular interest, in addition to the usual ones of consistency, independence and categoricity. The first of these may be termed "dimensional categoricity", by which is meant that all propositions whose hypotheses require the existence of at least, but not more than, n dimensions are demonstrable in terms of axioms which require the existence of at most n dimensions. The second property may be described roughly by saying that the space is characterized by its one dimensional axioms: that is to say that a set of axioms sufficient to establish fully a space of $n(>1)$ dimensions may be obtained from the one dimensional set by removing the one dimensional closure axiom and adding successive existence axioms and a new closure axiom.

The axioms of this set are stated in terms of subclasses of a single class of undefined elements, called *points*. The notation for a point is a single capital letter, as A , B , etc. Certain triads of points are assumed to be in an undefined relation, called *order*, the notation ABC (to be interpreted, "the points A , B , C are in the order ABC ") being used to indicate this relation. Certain segments (subclasses of points to be defined later) are assumed to be in the undefined relation of *congruence*, one segment being said to be congruent to another segment. The symbol $\equiv$ placed between the symbols for the two segments indicates the presence of this relation.

These are the only essentially geometrical notions postulated. The writer believes that the whole of the geometry may be constructed from these elements, using in addition the concepts and laws of pure logic. No satisfactory test of consistency and categoricity have yet been devised, however, which do not involve the assumption of the existence, consistency and categoricity of the number system of algebra. The question whether these latter ideas are themselves consequences of pure logic, without additional assumptions, is not raised in this paper, and free use will be made of algebraic concepts.

For the purposes of this article double elliptic geometry of n dimensions will be taken to be isomorphic with the set of all sets of $n+1$ real numbers

* Received April 13, 1927.



$(x_1, x_2, \dots, x_{n+1})$, subject to the condition that $x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1$, together with all propositions derivable from them by the operations of logic and algebra. In one and two dimensions this definition gives the geometry of the circle and of the sphere, respectively.

The axioms of the set are separated into four groups: (1) the order axioms, O, stated in terms of point and order only, and primarily concerned with the properties of the order relation; (2) the congruence axioms, C, stated in terms of point, order and congruence, and primarily concerned with the properties of the congruence relation; (3) the existence axioms, E, a succession of axioms postulating in turn the existence of the first, second and third dimensions; (4) the closure axioms, S, one for each dimension, each limiting the space to a particular dimensionality.

The segregation of the group *E* may seem rather artificial at first glance, since points are introduced into the geometry by others of the axioms, e. g. O2 and O6. It will be observed, however, that the axioms E are the only ones which, in the absence of all but those axioms which are necessary to satisfy the hypothesis, demand the existence of additional points.

References to supporting theorems, axioms etc. are enclosed in square brackets [] and are given by letter and number. The letter T is used for theorems, D for definitions, O, C, E and S, respectively, for order, congruence, existence and closure axioms. Frequent references to the axioms and theorems of J. R. Kline's article, "Double Elliptic Geometry in terms of Point and Order alone"*, have made it desirable to assign the symbols K-I, K-II etc. to his axioms, and K1, K2 etc. to his theorems.

The distinctness of points, or other entities, is not to be assumed unless explicitly stated, or unless implicit in the definition of a class to which the entities belong. Symbolic statements are to be read in the grammatical construction to which the context assigns them. Thus the statement "if ABC " may be interpreted "if A is a point, and B is a point, and C is a point, and if A, B, C are in the order ABC ". The symbol $\{P\}$ is used to denote a set of points P , particular members of the set being differentiated by subscripts. Except in algebraic equations, the symbol $=$ is used to indicate identity. A vertical line through a symbol of relation is to mean the negation of the relation.

2. Axioms and definitions. O1. If A is any point, and if there exists a point B , distinct from A , then at least one such point B is such that ABX is false for every X .†

D1. If A and B are such that $A \neq B$, and if ABX is false for every X , then B is said to be an *opposite* of A ($B = A'$). Wherever A' has

* *Annals of Math.*, (2) vol. 18, pp. 31-44.

† With reference to axioms O, E and S, cf. Kline, loc. cit.

not been specified to be a particular opposite of A , the symbols, $B \neq A'$, are to mean that B is distinct from *every* opposite of A . The points of a set $\{P\}$ will be said to be *non-opposite* if no one of them is an opposite of another.

O2. If there exist two distinct points, A and B , then at least two such points are such that B is not an opposite of A .

O3. If ABC , then CBA .

O4. If ABC , and if A' is an opposite of A , then $A'CB$.

D2. If A and B , and if $B \neq A'$, then the set of all points X such that $X = A$, or $X = B$, or AXB , is called the *segment* AB , having A and B as *endpoints*, and the set of points X such that AXB as *interior*. The symbol AB will mean the segment AB whenever it occurs without immediate qualification. In the presence of O3 and O4 (as will be seen later [T2]) $AB = BA$. If $B = A$, then AB is called a *null* segment; otherwise, a *proper* segment.

O5. If A, B, C, D, E are points such that $ABD, AEC, C \neq B', E \neq D'$, then either a point F on DE is such that BFC or a point G on BC is such that DGE .*

D3. Let AB be any segment. Let $\{S\}$ and $\{T\}$ be any two subsets of AB such that $\{S\} \neq 0, \{T\} \neq 0, \{S\} \cdot \{T\} = 0, \{S\} + \{T\} = AB$, and such that $S_1 T_0 S_2$ and $T_1 S_0 T_2$ are false for every $S_0, S_1, S_2, T_0, T_1, T_2$. If, for every two such sets, $\{S\}$ and $\{T\}$, there exists a point H such that SHT for every $S(\neq H)$ and for every $T(\neq H)$, then the segment AB is said to be *continuous*.

O6. If AB is any segment, then AB is continuous.

D4. If A and B are such that $A \neq B \neq A'$, then the set of all points X such that $X = A$, or $X = B$, or AXB , or ABX , is called the *ray* AB , *starting from* A . The ray AB plus the point A' is called the *halfline* ABA' .

C1^b. If AB is any segment and CD is any ray, then on ray CD there exists $\left\{ \begin{smallmatrix} \text{at least} \\ \text{not more than} \end{smallmatrix} \right\}$ one point E such that $AB \equiv CE$.†

C2. If A, B, C, D, E are points such that $A \neq C \neq A', D \neq E \neq D', B$ lies on $AC, B \neq C, AC \equiv DE$, and if there exists a point F on ray DE such that $AB \equiv DF$, then at least one such point F is on DE and distinct from E .

* Cf. G. H. Hallet, Jr., Linear order in three dimensional Euclidean and double elliptic spaces, *Annals of Math.* (2) vol. 21 (1921) p. 200.

† See R. L. Moore, Sets of metrical hypotheses for geometry, *Trans. Amer. Math. Soc.* vol. 9 (1908), pp. 487–512. Making due allowance for the different types of order in the two geometries, the following relations between his congruence axioms and ours may be observed: our C1 is analogous to his Theorem 1, which is a consequence of his C₁; our C2 contains another implication of his C₁; our C3 is necessitated by our different definition of segment and has no analogue in his set; our C4 is his C₂; our C5 combines his C₃ and C₄ (Cf. his note on C₄).

C3. If $A \nmid B \nmid A'$, then $AB \equiv AA$ is false.

C4. If $AB \equiv CD$ and $CD \equiv EF$, then $AB \equiv EF$.

C5. If A, B, C, D, E, F, G, H are points such that $ACD, D \nmid A', EGH, H \nmid E', B \nmid A', B \nmid C', D \nmid B \nmid D', F \nmid E', F \nmid G', H \nmid F \nmid H', AB \equiv EF, AC \equiv EG, AD \equiv EH, CB \equiv GF$, then $DB \equiv HF$.

E1. There exist at least two distinct points.

D5. If A and B are points such that $A \nmid B \nmid A'$, then the set of all points X such that $X = A$, or $X = B$, or $X = A'$, or $X = B'$, or XAB , or AXB , or ABX , or $AX'B$, is called the *line* AB . In the presence of O3 and O4 (as in D2), line $AB =$ line BA . The points of a set $\{P\}$ will be said to be *collinear* if there exists a line containing all of them.

E2. If there exist three distinct, non-opposite points, then there exist at least three such points A, B, C , such that C does not lie on line AB .

D6. If A, B, C are distinct and non-collinear, and if P and Q are distinct and non-opposite points of distinct segments AB, BC, CA , then the set of all points X such that X lies on some segment PQ is called the *triangle* ABC (or ACB etc.), having A, B, C as *vertices*, the interiors of the segments AB, BC, CA as *sides*, and the sum of the interiors of the segments PQ as *interior*. The set of all points X such that X lies on some line PQ , but not on the triangle ABC , is called the *exterior* of the triangle.

D7. If ABC is any triangle, then the triangle ABC plus its exterior is called the *plane* ABC . The points of a set $\{P\}$ will be said to be *complanar* if there exists a plane containing all of them.

E3. If there exist three distinct, non-collinear points, then there exist at least one plane ABC and one point D such that D does not lie on the plane ABC .

D8. If A, B, C, D are distinct and non-complanar, and if P and Q are distinct and non-opposite points of distinct triangles ABC, ABD, ACD, BCD , then the set of all points X such that X lies on some segment PQ is called the *tetrahedron* $ABCD$ (or $ABDC$ etc.), having A, B, C, D as *vertices*, the interiors of the segments AB, AC, AD, BC, BD, CD as *edges*, the interiors of the triangles ABC, ABD, ACD, BCD as *faces*, and the sum of the interiors of the segments PQ as *interior*. The set of all points X such that X is on some line PQ , but not on the tetrahedron $ABCD$, is called the *exterior* of the tetrahedron.

D9. If $ABCD$ is any tetrahedron, then the tetrahedron, $ABCD$ plus its exterior is called the *three-space* $ABCD$.

S1. If there exist points A and B such that $A \nmid B \nmid A'$, then at least two such points are such that every point lies on the line AB .

S2. If there exist three distinct, non-collinear points, then there exist three such points, A, B, C , such that every point lies on the plane ABC .

S3. If there exist four distinct, non-complanar points, then there exist four such points, A, B, C, D , such that every point lies on the three-space $ABCD$.

3. **One dimension.** T1. If A and B are points such that $A \neq B \neq A'$, then there exists a point C such that ABC [K1].

T2. If B is an opposite of A , then A is an opposite of B [K2].

T3. If ABC , then $C \neq A$ and $C \neq B'$ [Part of K3].

T4. If $ABC, ACD, D \neq B'$, then BCD [Weak form of K12].

Proof. $C \neq C'$ [D1]. $ABC, ACD, D \neq B', C \neq C'$ imply that some point F on CC is such that DFB [O5] since G on DB and such that CGC is impossible [T3]. F on CC implies CFC or $F = C$ [D2]. CFC is impossible [T3]. Hence $F = C$, i. e. BCD .

T5. If ABC , then $C \neq B$ and $C \neq A'$ [Remainder of K3].

Proof. (1) Suppose ABB . Then $B \neq B'$ [D1]; and $ABB, ABB, B \neq B'$, imply BBB [T4], which is impossible [T3]. (2) Suppose ABA' . Then $A'A'B$ [O4], and $BA'A'$ [O3], which is impossible by case 1.

T6. If ABC , then not ACB .

Proof. $B \neq B'$ [D1]. $ABC, ACB, B \neq B'$ imply BCB [T4], which is impossible [T3].

T7. If ABC and ACD , then BCD [K12 in full].

Proof. The theorem is immediate if it can be shown that $D = B'$ is impossible [T4]. Now $D = B'$ implies ACB' [hyp.], which implies $B'CA$ [O3], which implies BAC [T2, O4], which implies CAB [O3]. CAB implies not CBA [T6], which implies not ABC [O3], contradicting the hypothesis.

T8. If ABC and ACD , then ABD [K-V].

Proof. ABC implies CBA [O3], and ACD implies DCA [O3], $D'AC$ [O4], CAD' [O3]. Then CBA, CAD' imply BAD' [T7], which implies ABD [O3, T2, O4, O3].

It is easily seen that Kline's first two axioms are equivalent to O1, O2 and E1. His third and fourth axioms are identical with O3 and O4, respectively. His fifth axiom has been shown by T8 to be a consequence of certain axioms of our set. Since his first sixteen theorems are consequences of his first five axioms, we shall assume without further proof such of these theorems as have not already been proved explicitly.

The implications of ABC with respect to the points A, B, C, A', B', C' and the possible orders connecting these points are of frequent occurrence. For the sake of brevity we collect these statements in the next theorem and we shall use the theorem without explicit reference to it. The proof of the theorem will be omitted, since some parts of it have already been

demonstrated and the remainder are, in each case, almost immediate consequences of axioms and theorems already stated, or of simple combinations of them.

T9. ABC implies the distinctness of every two of the points A, A', B, B', C, C' ; ABC implies the truth of the orders $ABC, CBA, A'CB, BCA', BAC', C'AB, B'A'C, CA'B', AC'B', B'C'A, A'B'C', C'B'A'$; ABC implies the falsity of every other order connecting points selected from A, A', B, B', C, C' .

T10. If AB is any ray and $C(\neq A)$ is any point on ray AB , then B is on ray AC .

Proof. This is evidently an immediate consequence of D4.

T11. If C is on ray AB , then C' is not on ray AB .

Proof. C on ray AB implies $C = A$, or $C = B$, or ACB , or ABC ; C' on ray AB implies $C' = A$, or $C' = B$, or $AC'B$, or ABC' [D4]. Comparison with D4 and T9 shows that no one of the first four statements is compatible with any one of the second four.

T12. If AB is any segment, then $AB \equiv AB$.*

Proof. (1) Suppose AB is a null segment, AA . Let K be any point such that $A \neq K \neq A'$ [K 6]. Then on ray AK there exists C such that $AA \equiv AC$ [C1a]. Suppose $C \neq A$. Then ACK , or $C = K$, or AKC , any one of which implies $C \neq A'$. Hence there exists ray AC , and on it a point D such that $AC \equiv AD$ [C1a] and such that $D \neq A$ [C3]. Hence $AA \equiv AD$ [C4] and $C = D$ [C1b], i. e. $AC \equiv AC$. If we now identify the points A, B, C, D, E of C2 with the points A, A, C, A, C , respectively of this theorem, we see that the hypothesis of this axiom is satisfied in such a way that only the point C on ray AC is such that $AA \equiv AC$, and therefore $C \neq C$, which is impossible. Hence $C = A$, i. e. $AA \equiv AA$.

(2) Suppose AB is a proper segment. Then there exists $C(\neq A)$ on ray AB such that $AB \equiv AC$ [C1a, C3]. As in case 1, it can be shown that $AC \equiv AC$. Then $C(\neq A)$ on ray AB implies ACB , or ABC , or $C = B$. Either ACB or ABC implies $C \neq B$. If ACB , then by identifying the points A, B, C, D, E of C2 with the points A, C, B, A, C of this theorem, it results (as in case 1) that $C \neq C$. The order ABC gives rise in a similar manner to the same contradiction. Hence $C = B$, i. e. $AB \equiv AB$.

T13. If $AB \equiv CD$, then $CD \equiv AB$.

Proof. Let K be a point such that $A \neq K \neq A'$ and such that B lies on ray AK . (If AB is a proper segment, let $K = B$). Then there exists E on ray AK such that $CD \equiv AE$ [C1a]. $AB \equiv AE$ [C4] and $AB \equiv AB$ [T12]. Hence $B = E$ [C1b], i. e. $CD \equiv AB$.

* Several of the next few theorems, which deal with the properties of the congruence relation, are the counterparts of theorems of Moore, loc. cit.

T14. If $AB \equiv CD$ and $AB \equiv EF$, then $CD \equiv EF$.

Proof. This follows at once from T13 and C4.

T15. If AB is any proper segment and CC is any null segment, then $AB \not\equiv CC$ and $CC \not\equiv AB$.

Proof. Suppose $AB \equiv CC$. Let K be any point such that $C \neq K \neq C'$, and let D be any point on the interior of AB . Then there exists exactly one point E on ray CK such that $AD \equiv CE$. Now $E \neq C$, since $E = C$ implies $AD \equiv AB$ [T13, C4] and (since $AD \equiv AD$) $D = B$, contrary to the assumption that D was interior to AB . Then $CE \equiv AD$ and $CC \equiv AB$; hence B lies on the segment AD and is distinct from D [C2]. This implies either $B = A$, contrary to hypothesis, or ABD , which contradicts the assumption that D was on AB . Hence $AB \not\equiv CC$.

$CC \not\equiv AB$ is an immediate consequence of this and T13.

T16. If AA and BB are null segments, then $AA \equiv BB$.

Proof. Let K be any point such that $B \neq K \neq B'$, and let C be the point on ray BK such that $AA \equiv BC$. $C \neq B$ is impossible [T15]. Hence $AA \equiv BB$.

D10. If AB and CD are segments, and if there exists $E(\neq D)$ on CD such that $AB \equiv CE$, then AB is said to be *less than* CD ($AB < CD$), and CD is said to be *greater than* AB ($CD > AB$).

T17. If AB and CD are any segments, then exactly one of the statements, $AB < CD$, $AB \equiv CD$, $AB > CD$, is true.

Proof. (1) Suppose that both AB and CD are null segments. Then $AB \equiv CD$ [T16]. Neither $AB < CD$ nor $AB > CD$ is possible [D10]. Hence only the statement $AB \equiv CD$ is true.

(2) Suppose AB is a null segment and CD is a proper segment. Then C is the only point E on ray CD such that $AB \equiv CE$ [T16, C1]. Since C is on CD and $C \neq D$, $AB < CD$ [D10]. $AB \equiv CD$ is impossible [T15]. $AB > CD$ is impossible [D10, T3, T15]. Hence only the statement $AB < CD$ is true.

(3) Suppose AB is a proper segment and CD is a null segment. By reasoning similar to that in case 2, only the statement $AB > CD$ is true.

(4) Suppose both AB and CD are proper segments. Let $E(\neq C)$ be the point on ray CD such that $AB \equiv CE$. Then exactly one of the statements CED , $E = D$, CDE is true [D4]. (a) If CED , then $AB < CD$. $AB \equiv CD$ is impossible, since it would imply $D = E$, which contradicts CED . If $AB > CD$, then there exists an interior point F of AB such that $CD \equiv AF$ [D10, T15]; AFB , $AF \equiv CD$, $AB \equiv CE$, D on ray CE would imply CDE [C2], contrary to the assumption CED . Hence CED implies only $AB < CD$. (b) If $E = D$, then $AB \equiv CD$. $AB < CD$ and $AB > CD$ are evidently impossible [C1]. (c) If CDE , then there exists

exactly one point F on ray AB such that $CD \equiv AF$, and F is interior to AB . As in case (a), only the statement $CD < AB$, i. e. $AB > CD$, is true.

T18. If APB and AQB , then APQ , or $P = Q$, or AQP [K25].

Proof. $A \nmid P \nmid B$ and $A \nmid Q \nmid B$. Exactly one of the statements, $AP < AQ$, $AP \equiv AQ$, $AP > AQ$, is true [T17]. If $AP < AQ$, then there exists just one point R such that ARQ and $AP \equiv AR$. ARQ and AQB imply ARB [T7], i. e. R on ray AB . Hence $P = R$ [C1], i. e. APQ .

If $AP \equiv AQ$, then $P = Q$.

If $AP > AQ$, then $AQ < AP$ and, by reasoning similar to above, AQP .

T19. If ABC and ABD , then ACD , or $C = D$, or ADC [K24].

Proof. ABC implies $A'CD$, ABD implies $A'DB$. Hence $A'DC$, or $C = D$, or $A'CD$ [T18]. $A'DC$ implies ACD , and $A'CD$ implies ADC .

T20. If ABC and ABD , then BCD , or $C = D$, or BDC [K23].

Proof. This follows easily from T19 and T7.

T21. If ABC and BCD , then ABD , or $D = A'$, or $AB'D$ [K22].

Proof. Since ABC implies BCA' , this theorem is a ready consequence of T19.

T22. If ABC and BCD , then D is on line AB [K17].

Proof. BCD implies CDB' , ABC implies CBA . CBD' and CBA imply BAD' , or $BD'A$, or $D' = A$ [T29]. Hence ABD , or BDA' , or $D = A'$; i. e. D lies on line AB [D5].

The proofs of K18 and K19 are based on Kline's first five axioms and on his Theorem 17; they do not involve his Axiom VII explicitly. Since we have now demonstrated K17 [T22], we shall assume the truth of K18 and K19.

Kline's axiom X (a form of the Dedekind cut postulate) is a consequence of D3 and O6, since every segment in his sense is a segment in our sense.

We have now assumed or proved each of Kline's specifically one dimensional axioms and theorems which are stated in terms of point and order. However, since the existence of congruence relations in his geometry depends upon three dimensional concepts, we shall not now attempt to show separately that every theorem in one dimension in terms of point and order which is a consequence of his set of axioms is a consequence of this set. That will follow immediately when this set has been established as categorical.

T23. If ABC , DEF , $AB \equiv DE$, $AC \equiv DF$, then $BC \equiv EF$.

Proof. ABC implies $C \nmid A'$, $B \nmid C$, $B \nmid A'$, $B \nmid C'$. Similarly DEF implies $F \nmid D \nmid E$ and $F \nmid E \nmid F'$. $BB \equiv EE$ [T14]. If now the points A, B, C, D, E, F, G, H of C5 are identified with the points A, B, B, C, D, E, E, F of this theorem, the conditions of the hypothesis of C5 are

fulfilled and we have $CB \equiv FE$. Since $CB = BC$ and $FE = EF$ [D2], $BC \equiv EF$.

The theorem just proved constitutes a specialization of C5 which is more readily applicable to one dimensional theorems.

T24. If $ABC, DEF, AB \equiv DE, BC \equiv EF$, then $AC \equiv DF$.

Proof. The point G on ray DF such that $AC \equiv DG$ [C1] is such that DEG [C2]. Also, $BC \equiv EG$ [T23] and $EF \equiv EG$ [T14]. DEF and DEG imply G on ray EF [T20]. Hence $F = G$, i. e. $AC \equiv DF$.

T25. If $AB \equiv CD$, then $A'B \equiv C'D$.

Proof. If $A'B < C'D$, there exists E such that $A'B \equiv DE$ and DEC' , i. e. such that CDE . Then there exists F such that $CE \equiv AF$ and ABF . Then $BF = DE$ [T24] and $BF \equiv BA'$, i. e. $F = A'$, which contradicts ABF . Similarly, $A'B > C'D$ is impossible.

It is now possible to develop a theory of addition and subtraction of segments, either preserving the restrictions imposed by our type of order, or by extending order and congruence by definition. It is also possible to prove the various theorems of the type, "If $AB < CD$ and $CD < EF$, then $AB < EF$ ", and "If $AB < CD$ and $CD \equiv EF$, then $AB < EF$ " etc.

D11. If AB is any segment and C is a point of AB such that $AC \equiv BC$, then C is called a *midpoint* of AB . If ABA' is a half-line and C is a point of ABA' such that $AC \equiv A'C$, then C is said to be a *midpoint* of ABA' , and AC and $A'C$ are called *quadrants*.

T26. If AB (or ABA') is any segment (or half-line), then there exists exactly one midpoint of AB (or of ABA').

Proof. If AB is a null segment, the point A is evidently a midpoint, and the only midpoint, of AB . In the case of a proper segment AB or a half-line ABA' , the proof of the existence of a midpoint involves a familiar type of argument based on O6, and differs only in detail from R. L. Moore's proof of his Theorem 1, section 5*. The uniqueness of the midpoint is evidently assured by C2.

T27. If $AB \equiv CD$, and if E and F are the midpoints of AB and CD respectively, then $AE \equiv CF$.

T28. All quadrants are congruent.

The truth of these theorems follows from C2.

The space under consideration has now been developed to a point where it seems unnecessary to carry detailed investigation further. With the help of the continuity axiom [O6], it is possible to assign to each point P of our space a distinct real number α ($0 \leq \alpha < 2\pi$) in such a way that (1) to each real number α ($0 \leq \alpha < 2\pi$) there corresponds exactly one point P ;

* R. L. Moore, loc. cit., p. 499.

(2) points P_1, P_2, P_3 are in the order $P_1 P_2 P_3$ if, and only if, there exist numbers $\beta_1, \beta_2, \beta_3$ ($\beta_i = 0, 1$) such that either (a) $\alpha_1 + 2\beta_1\pi < \alpha_2 + 2\beta_2\pi < \alpha_3 + 2\beta_3\pi$ and $\alpha_3 + 2\beta_3\pi - (\alpha_1 + 2\beta_1\pi) < \pi$ or (b) $\alpha_1 + 2\beta_1\pi > \alpha_2 + 2\beta_2\pi > \alpha_3 + 2\beta_3\pi$ and $\alpha_1 + 2\beta_1\pi - (\alpha_3 + 2\beta_3\pi) < \pi$; (3) $P_1 P_2 \equiv P_3 P_4$ if, and only if, one of the quantities, $\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 2\pi + (\alpha_1 - \alpha_2), 2\pi + (\alpha_2 - \alpha_1)$, is equal to one of the quantities $\alpha_3 - \alpha_4, \alpha_4 - \alpha_3, 2\pi + (\alpha_3 - \alpha_4), 2\pi + (\alpha_4 - \alpha_3)$. This procedure is equivalent to establishing one dimensional double elliptic geometry, as defined in the first section. This constitutes a demonstration of the consistency and categoricity of the set of axioms composed of O, C, E1, S1.

The independence of the set will be established in the usual manner by exhibiting, for each axiom X_n , a geometry GX_n in which axiom X_n is false and the remaining axioms are true. Detailed proof of the validity of the examples is omitted.

GO 1. The set of all points in space is the set of all points on a Euclidean straight line; order and congruence have their usual significance.*

GO 2. There exist only the points Q and R ; every order is false; every segment is congruent to every segment.

GO 3. There exist exactly three points, Q, R, S ; only the orders QQS, QSS, RSQ, RSS, SSS are true; every null segment is congruent to every null segment, every proper segment is congruent to every proper segment, but no other congruence relations are true.

GO 4. There exist exactly three points, Q, R, S ; only the orders QRS and SRQ are true; congruence is defined as in GO 3.

GO 5. There exist exactly four points, P, P', Q, Q' ; only the orders $PPQ, QPP, P'P'Q, QP'P', PPQ', Q'PP, P'P'Q', Q'P'P', PQQP', P'QP', PQ'P', P'Q'P$ are true; congruence is defined as in GO 3.

GO 6. The set of all points is the set of all points on the unit circle each of which can be obtained from the point (1,0) by a rotation about the center of the circle through an angle which is a rational multiple or sub-multiple of 2π .

In the examples for C1-C5 the set of all points is the set of all points lying on a circle; order has its usual significance for points lying on an arc subtending a central angle less than π .

GC 1a. $AB \equiv CD$ if, and only if, the following conditions are satisfied simultaneously: (1) the segments are congruent in the ordinary sense; (2) a particular point O is not an endpoint of either segment.

GC 1b. Every null segment is congruent to every null segment, every proper segment is congruent to every proper segment, but no other congruence relations are true.

* Many of these examples will be found to be identical with, or slight modifications of, examples to be found in the papers of R. L. Moore and Kline already referred to.

GC 2. $AB \equiv CD$ if, and only if, CD is congruent in the ordinary sense to a particular segment PQ .

GC 3. Every segment is congruent to every null segment.

GC 4. $AB \equiv CD$ if, and only if, the central angle subtended by CD is equal to $\frac{2}{3}\pi$ plus one-third of the central angle subtended by AB .

GC 5. In order to define congruence three points, Y_1, Y_2, Y_3 , are selected at intervals of 120° along the circle. Let $l(AB)$ be the length of the shorter arc AB , measured in terms of 120° of arc as a unit. Let $m(AB)$, called the measure of AB , be defined as follows: (1) If AB lies wholly on one of the shorter arcs Y_1Y_2, Y_2Y_3, Y_3Y_1 , then m is given as a function of l by the relations

$$0 \leq l \leq \frac{1}{2}, \quad m = \frac{1}{2}l \quad \text{and} \quad \frac{1}{2} \leq l \leq 1, \quad m = \frac{1}{2}(3l - 1)$$

and l is given as a function of m by the relations

$$0 \leq m \leq \frac{1}{4}, \quad l = 2m \quad \text{and} \quad \frac{1}{4} \leq m \leq 1, \quad l = \frac{1}{3}(2m + 1);$$

(2) if AB lies so that any of the points Y_i separate it into distinct subsegments so that each of these subsegments lies wholly on one of the shorter arcs Y_iY_j , then the measure of AB is taken as the sum of the measures of the subsegments.

Then $AB \equiv CD$ if, and only if, $m(AB) = m(CD)$.

GE 1. There exists only one point; every order is false; there is no congruence.

GS 1. Double elliptic geometry of two dimensions.

4. **Two dimensions.** T 29. Two distinct lines can have at most a point and its opposite in common [K 20].

Proof. As in the case of K 18 and K 19 (v. section 3), this theorem is a consequence of Kline's Axioms I-V and his Theorem 17, so that his proof may be accepted.

T 30. If AB is any line and C is any point not on line AB , then there exist a unique line AC and a unique line BC , and lines AC and BC are distinct from each other and from line AB .

Proof. The existence and uniqueness of lines AC and BC are assured by $A \neq C \neq A'$ and $B \neq C \neq B'$ [D 5]. The distinctness of lines AC, BC and AB is assured by the fact that C is not on line AB [K 19].

T 31. If D is not on line AB , and if C and E are such that ABC and BED , and if there exists F such that DFA , then there exists G such that CEG and DGA [K-VI].

Proof. There exist distinct lines AB, AD, BD [T 30]. ABC implies BAC' . D not on line AB implies $D \neq A'$ [D 5, T 29]. BED implies E on line BD

and $B \neq E \neq D'$ and these, together, imply $C \neq E \neq C'$ [T 29, T 30]. BAC' , BED , $D \neq A'$, $E \neq C$ imply the existence of G on $C'E$ such that AGD [C 5].

G on $C'E$ implies $G = C'$, or $G = E$, or $C'GE$. But $G = C'$ implies G on line AB , while AGD implies G on line AD , i. e., since lines AB and AD are distinct, $G = A$, or $G = A'$; either of which contradicts AGD . Similarly, $G = E$ implies $G = D$ or $G = D'$, either of which contradicts AGD . Hence $C'GE$ and AGD , i. e. CEG and DGA .

We have shown in the preceding section that Kline's first five axioms are consequences of certain of our one dimensional axioms (not including S 1); his Axiom VI has just been proved [T 31]; his Axiom VII appears in our set under different wording as E2. Hence we shall accept his Theorems 1-36, which are based on his Axioms I-VII, as consequences of our set.

D 12. Let AB be a line and C a point not on the line. Then C plus the set of all interior points of segments having C as one endpoint and a point of line AB as the other endpoint, constitute the *half-plane* $C-AB$. It is evident from D 7, K 34, K 35 and K 36, that any line of a plane separates the plane into exactly two half-planes, and that the half-plane $C-AB$ is identical with the half-plane $D-AB$, where D is any point of $C-AB$. Also, by applying K30 and D5 to the triangles into which a point D of $C-AB$ divides $C-AB$, we may conclude that a point and its opposite do not lie in the same half-plane.

T 32. If ABC and EFG are triangles such that $AB \equiv EF$, $AC \equiv EG$, $CB \equiv GF$, and if D and H are points of rays AC and DG , respectively, and such that $AD \equiv EH$, then $BD \equiv FH$.

Proof. By hypothesis, $D = A$, or $D = C$, or ADC , or ACD . If $D = A$ or $D = C$, then $H = E$ or $H = G$, respectively [C 1], and the theorem is trivial. If ACD , then EGH [C 2], and the theorem is a special case of C 5. If ADC , then EGH ; $AC \equiv EG$ implies $A'C \equiv E'G$ and $AB \equiv EF$ implies $A'B \equiv E'F$ [T 25]. Also, $A'CD$ and $E'GH$. The theorem again reduces to C 5.

D 13. If ABC is any triangle, then the rays AB and AC are said to form the *angle* BAC ($\angle BAC$), whose *vertex* is A , and whose *sides* are the rays AB and AC without the vertex. Those points of the plane ABC which are common to the half-planes $B-AC$ and $C-AB$ form the *interior* of $\angle BAC$, and those points of the plane ABC which are neither on the angle BAC nor in its interior, except A' , form its *exterior*. The following is evidently an equivalent definition of the exterior and interior of the angle: if P and Q are non-opposite points of distinct sides of the angle, then the set of all points interior to some PQ form the interior of the angle, and the set of all points on some line PQ , but not on the segment PQ , form the exterior of the angle.

D 14. $\angle BAC$ is said to be *congruent* to $\angle EDF$ ($\angle BAC \equiv \angle EDF$) if there exist points K, L, M, N such that K and L lie on sides AB and AC , respectively, of $\angle BAC$, M and N lie on sides DE and DF , respectively, of $\angle EDF$, $AK \equiv DM$, $AL \equiv DN$, and $KL \equiv MN$.

T 33. If ABC and DEF are triangles such that $\angle BAC \equiv \angle EDF$, $AB \equiv DE$, $AC \equiv DF$, then $BC \equiv EF$, $\angle ABC \equiv \angle DEF$ and $\angle ACB \equiv \angle DFE$.

Proof. Let K, L, M, N be points (v. D 14) by virtue of which $\angle BAC \equiv \angle EDF$. By T 32, $KC \equiv MF$, and, applying the theorem again, $BC \equiv EF$. Then $\angle ABC \equiv \angle DEF$ and $\angle ACB \equiv \angle DFE$ [D 14].

T 34. If BAC is any angle and $D-EF$ is any half-plane, then in $D-EF$ there exists one and only one ray EG , starting from E and such that $\angle BAC \equiv \angle GEF$.

Proof. Let P be the midpoint of ABA' and Q be the midpoint of ACA' . Let R be the midpoint of EDE' and S be the midpoint of EFE' . R and S are distinct and non-opposite [D 12]. On ray SR let T be the point such that $QP \equiv ST$. In triangles ESR and $E'SR$, $ES \equiv E'S$ and $ER \equiv E'R$ [T 25], and $SR \equiv SR$. Also, T lies on ray SR and $ST \equiv ST$. Hence $ET \equiv E'T$ [T 32]. In triangles EST and AQP , $ES \equiv AQ$ and $ET \equiv AP$ [T 25], and $ST \equiv QP$. Hence $\angle BAC \equiv \angle TEF$.

Suppose there exists H , not on ray ET , in $D-EF$ and such that $\angle BAC \equiv \angle HEF$. Ray EH , except for E , lies in $D-EF$, and line EH lies in plane DEF . Line EH intersects line RS , which also lies in plane ABC [K 35], in two opposite points, J and J' . One of these points, say J , lies in $D-EF$. Then $EJ \equiv E'J$ (as above for $ET \equiv E'T$). Hence, in triangles AQP and ESJ , $AQ \equiv ES$, $AP \equiv EJ$, $\angle PAQ \equiv \angle JES$, and therefore $PQ \equiv JS$. But since J lies in $D-EF$ and on line SR , it lies on ray SR , i.e. $J = T$. Hence there is one and only one ray satisfying the conditions of the theorem.

In the chapter on pure spherics in his "Rational geometry",* Halsted develops two dimensional double elliptic geometry rather fully. His set of axioms is consistent and, except for the lack of an explicitly stated closure axiom, categorical. In order to demonstrate the consistency and categoricity of our set of axioms, O, C, E 1, E 2, S 2, we shall show that his undefined notions may be defined in terms of ours in such a way that each of his axioms is a logical consequence of our set.

He assumes as undefined the notions of point, straightest and sphere. These we define as identical with our point, line and plane, respectively. The implicit assumption is made that all points and straightests lie on a single sphere; the truth of this assumption in our geometry is assured by

* G. B. Halsted, Rational geometry, Chapter 15.

K 35 and S 2. The opposite of a point A is defined as a point A' ($\neq A$) which with A does not determine a straightest; in our geometry the opposite of a point is a point, and the only point, having this property. All the axioms of his first group, the "assumptions of association on the sphere", now follow at once: I 1' is equivalent to O 1; I 2' is a part of D 5; the first part of I 3' (which is superfluous in his set) follows from K 19; the second part of I 3' is implicit in D 5; I 4' is equivalent to E 1, O 2, E 2.

The undefined relation of betweenness is next introduced. We may define this relation as follows: a point B is said to lie between points A and C if, and only if, ABC or CBA . He defines a sect as a pair of distinct non-opposite points, the points lying between these points being said to be the points of the sect. A sect is, then, our proper segment. Under his definition a ray is equivalent to our ray minus the starting point. Of his second set of axioms, the "assumptions of betweenness on the sphere", II 1', II 2', II 4', II 6' and II 8' are special cases of T 9; II 5' is T 1; II 3' is a consequence of T 1 and O 3; II 7' is T 8; II 9' is a consequence of K 28 and K 36; and II 10' is a consequence of D 4 and D 5.

Halsted then introduces the undefined relation of congruence between sects. We define sect AB congruent to sect CD if, and only if, segment AB is congruent to segment CD . The points which lie on one side of a line in his terminology form what we call a half-plane. His definition of an angle and of the exterior and interior of an angle are fully equivalent to ours. He also assumes the relation of congruence between angles; this we may define as in D 13. We can now show at once that the axioms of his last set, the "assumptions of congruence on the sphere", are also consequences of our set. The first part of III 1' is a re-wording of C 1 as applied to proper segments; $AB \equiv AB$ is T 10, while $AB \equiv BA$ is superfluous. III 2' is T 22. III 3' contains the implicit assumption that if points A, B, C, D, E, F are such that $ABC, D \neq E \neq D', F$ on ray ED' , $AB \equiv DE, BC \equiv EF$, then F is such that DEF . The proof of T 21 demonstrates the truth of this assumption, and the axiom then becomes a restatement of T 21. The first part of III 4' is equivalent to T 34, the location of the inner points of the angle being assured by D 13 and C 2; under our definition of congruence of angles, the latter part of III 4' follows from the corresponding propositions for segments. III 5' is an immediate consequence of T 22. III 6' is equivalent to T 36. This completes the list of Halsted's assumptions.

As in the one dimensional case independence will be demonstrated by examples. Since in nearly every case these examples are merely slight modifications of those used in the preceding section, only the modifications will be cited wherever possible.

GO 1 is a Euclidean plane.

GO 2-GO 5 are unchanged.

GO 6 is obtained from the corresponding one dimensional example by substituting "sphere" for "circle" and "point (1, 0, 0)" for "point (1, 0)".

GC 1-GC 5 are obtained from the corresponding one dimensional examples by substituting "sphere" for "circle" and "arc of a great circle" for "arc".

GE 1 remains unchanged.

GE 2. One dimensional double elliptic geometry.

GS 2. Three dimensional double elliptic geometry.

5. Three dimensions. We may take the geometry resulting from Kline's full set of axioms as a representation of this geometry. We have already shown that Kline's Axioms I-VII and X are consequences of our Axioms O, C, E 1 and E 2. His Axioms VIII and IX are identical in content with E 3 and S 3, respectively. Hence we may assert that the set of Axioms O, C, E and S 3 are sufficient to determine three dimensional double elliptic geometry, i. e. this set is consistent and categorical.

The independence examples of the preceeding section for O, C, E 1, E 2 may evidently be modified to serve for three dimensions in a manner exactly analogous to that in which the examples in section 3 for O, C, E 1 were modified to suit two dimensions. We may take GE 3 and GS 3 as two and four dimensional double elliptic geometry, respectively.

It seems clear that the number of dimensions may be extended indefinitely by the removal of S 3 and the addition of suitable existence and closure axioms.

In the preparation of this paper the writer has encountered certain problems which he has not yet solved. These relate to the further subdivision of certain of the axioms here presented. For example, each of the axioms, O 5 and C 5, might be subdivided into two parts, one applying to one dimension and the other to two dimensions. The writer has constructed a consistent and categorical set based on this modification of the present set, but he has not been able to settle the independence of the two dimensional form of O 5. Axiom O 6 might also be subdivided by assuming continuity for a single segment only and adding some form of Archimedean axiom. This can be done for the one dimensional set without sacrificing independence, but the writer has not found a two dimensional geometry of the double elliptic type in which the Archimedean axiom holds but the continuity axiom does not.

In conclusion the writer wishes to thank most sincerely Professor J. R. Kline, whose advice and encouragement have been invaluable in the preparation of this paper.

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